ON THE SYMMETRY CLASSES OF TENSORS ASSOCIATED WITH CERTAIN FROBENIUS GROUPS

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Abstract. In this paper, we discuss the existence of an orthogonal basis consisting of decomposable vectors for some symmetry classes of tensors associated with certain Frobenius subgroups of the full symmetric group.

AMS Classification: 20C30, 15A69.

Keywords: Symmetry classes of tensors, Frobenius groups.

1. Introduction

Let $n \geq 2$ and m be positive integer numbers. Denote by S_n the symmetric group on $\{1, 2, \ldots, n\}$. Let V be a unitary complex vector space of dimension m. Let $\otimes^n V$ be the n-th tensor power of V, and write $v^{\otimes} := v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for the tensor product of the indicated vectors.

For $\sigma \in S_n$, there is a unique linear operator $P(\sigma^{-1})$ on $\otimes^n V$ which has the effect $P(\sigma^{-1})(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$, for all $v_1, v_2, \ldots, v_n \in V$. Let G be a subgroup of S_n and χ be an irreducible complex character of G and $I(G)$ be the set of all the irreducible complex characters of G. We define $T(G, \chi)$ as a linear operator on $\otimes^n V$ with the following definition

$$T(G, \chi) := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma). \tag{1}$$

With respect to the induced inner product in $\otimes^n V$, $T(G, \chi)$ is an orthogonal projection onto its range $V_\chi^n(G)$, which is called the symmetry class of tensors associated with G and χ, and the dimension of $V_\chi^n(G)$ is (see [3], [7])

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) m^{c(\sigma)}, \tag{2}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ, (see [6]). It follows from the orthogonality relations for characters that $\{T(G,\chi) | \chi \in I(G)\}$ is a set of annihilating idempotents which sum to the identity. With respect to the induced inner product in $\otimes^n V$, and the orthogonal relations for characters we have:

$$\otimes^n V = \sum_{\chi \in I(G)} V_\chi^n(G) \tag{3}$$

which is an orthogonal direct sum.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $1 \leq \alpha_i \leq m$. Then the group G acts on Γ_m^n by $\sigma \cdot \alpha := \alpha \circ \sigma^{-1}$, $\sigma \in G$, which is a composition of two functions σ^{-1} and α.

Let Δ be a system of distinct representatives of the orbits of G acting on Γ_m^n and define:

$$\overline{\Delta} = \{\alpha \in \Delta | \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\}, \tag{4}$$

which $G_\alpha := \{\sigma \in G | \sigma \cdot \alpha = \alpha\}$ is the stabilizer of α. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of V and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Gamma_m^n$, the image of $e_\alpha^\otimes := e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n}$ under $T(G,\chi)$ is denoted by $e_\alpha^\chi := e_{\alpha_1} * e_{\alpha_2} * \cdots * e_{\alpha_n}$. For $\gamma \in \overline{\Delta}$, $V_\gamma^\chi = \langle e_{\sigma \cdot \gamma}^\chi | \sigma \in G \rangle$ is called the orbital subspace of $V_\chi^n(G)$. In [3], prove that

$$\dim V_\gamma^\chi = \frac{\chi(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \chi(\sigma) \tag{5}$$

Definition 1.1. An orthogonal basis of the form $\{e_\gamma^\chi | \gamma \in B\}$, where B is a subset of Γ_m^n, is called an orthogonal basis of decomposable symmetrized tensor for $V_\chi^n(G)$, in this case we say that $V_\chi^n(G)$ has an O-basis.

Since $V_\chi^n(G) = \sum_{\gamma \in \overline{\Delta}} V_\gamma^\chi$, so $V_\chi^n(G)$ has an O-basis if and only if V_γ^χ has an O-basis.

Several papers are devoted in investigation of the existence of an O-basis for $V_\chi^n(G)$, for example [8, 1, 2, 4]. In this paper we study the symmetry classes of tensors associated with some Frobenius groups of order pq, where $q|p-1$. We investigate the problem of finding necessary and sufficient conditions for the existence of an O-basis for the above mentioned groups.

2. The Group $F_{p,q}$

Let S be a faithful transitive G-set with each nontrivial element $g \in G$ having at most one fixed point. If no such g has a fixed point, then S is a regular G-set; if some g does have a fixed point, then G is called a Frobenius group.

Definition 2.1. [5] If p is a prime and $q|p-1$, then we write $F_{p,q}$ for the group of order pq with presentation

$$F_{p,q} = \langle a, b \ : \ a^p = b^q = 1, \ b^{-1}ab = a^u \rangle, \tag{6}$$

where u is an element of order q in the product group the integer numbers modulo p, Z_p^*.

The groups $F_{p,q}$ belong to a wider class of Frobenius groups. This group is of order pq and its elements are of the form $F_{p,q} = \{a^i b^j | 0 \leq i \leq p-1, \ 0 \leq j \leq q-1\}$. It is not hard to see that $F_{p,q}$ has $\frac{p-1}{q} + q$ conjugacy classes which are

$$\{1\}, \ \{a^{v_i s} | s \in S\}, \ i = 1, 2, \ldots, \frac{p-1}{q}, \ \{a^m b^l | m = 0, 1, \ldots, p-1\}, \ l = 1, 2, \ldots, q-1,$$

where S is the subgroup of Z_p^* generated by u, an element of order q in Z_p^* and $\{v_1, v_2, \ldots, v_{\frac{p-1}{q}}\}$ is a transversal set of S in Z_p^*, and the irreducible complex characters table of $F_{p,q}$ is:

(7)

σ	1	a^{v_i}	b^l
χ_k	1	1	ω^{lk}
φ_j	q	$\sum_{s \in S} \omega^{\frac{q}{p} s v_i v_j}$	$0,$

where $\omega = exp(\frac{2\pi i}{q}), \ 0 \leq k \leq q-1, \ 1 \leq l \leq q-1, \ 1 \leq i, j \leq \frac{p-1}{q}$.
From the above table we see that $F_{p,q}$ has q linear characters $\chi_k, \ 0 \leq k \leq q-1$, and $\frac{p-1}{q}$ non-linear $\varphi_j, \ 1 \leq j \leq \frac{p-1}{q}$ of degree q. Now we will embed this group in a suitable symmetric group. We can consider $(1 \ 2 \ 3 \ \cdots \ p)$ is a permutation in S_p, it can be verified that the mapping $a \mapsto (1 \ 2 \ 3 \ \cdots \ p)$, since $b^{-1}ab = a^u \mapsto$

$(\ 1 \ \ 1+u \ \ 1+2u \ \cdots \ 1+(p-1)u \ )$, so b can be map to the following permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & \text{p} \\ 1 & \text{1+u} & \text{1+2u} & \cdots & \text{1+(p-1)u} \end{pmatrix}$$

in S_p. So $F_{p,q}$ can be consider as a permutation subgroup in S_p, i.e., $F_{p,q} \leq S_p$.

Theorem 2.1. Let p be a prime number and $G = F_{p,p-1}$. We have only one non-linear irreducible character φ, where $\varphi(1) = p-1$, $\varphi(a) = -1$ and $\varphi(b^l) = 0$, $l = 1, 2, \dots, p-2$. Let V be an m-dimensional inner product space, then the dimension of the symmetry classes of tensor associated with G and φ is

$$dim V_{\varphi}^{p}(G) = \frac{p-1}{p}(m^p - m)$$

Corollary 2.1. (little Fermat theorem) Let p be a prime and m be a positive integer number, then $m^p \equiv m(modp)$.

Proof. By using Theorem 2.2, since $\frac{p-1}{p}(m^p - m)$ is an integer number, so $p|m^p - m$. □

In general, the mapping from $F_{p,q}$ to S_P is not known, therefore we can't calculate the dimension of the symmetry classes of tensor associated with $F_{p,q}$ and linear irreducible character, in the next example we calculate these for $F_{5,4}$.

Example 2.1. If (1 2 3 4 5) and (2 3 5 4) are permutations in S_5, then it can be verified that the mapping $a \mapsto (1\ 2\ 3\ 4\ 5)$, $b \mapsto (2\ 3\ 5\ 4)$, embeds $F_{5,4}$ in S_5. Now considering $F_{5,4}$ as a subgroup of S_5. By use (7) the character table of $F_{5,4}$, we find the dimensions of the symmetry classes of tensors associated with this group as follows:

$$\begin{aligned}
\text{d}im\ V_{\chi_0}^5 &= \tfrac{1}{20}[m^5 + 5m^3 + 10m^2 + 4m] \\
\text{d}im\ V_{\chi_1}^5 &= \tfrac{1}{20}[m^5 - 5m^3 + 4m] \\
\text{d}im\ V_{\chi_2}^5 &= \tfrac{1}{20}[m^5 + 5m^3 - 10m^2 + 4m] \\
\text{d}im\ V_{\chi_3}^5 &= \tfrac{1}{20}[m^5 - 5m^3 + 4m] \\
\text{d}im\ V_{\varphi}^5 &= \tfrac{4}{5}[m^5 - m]
\end{aligned}$$

Corollary 2.2. Let m be a positive integer number, then the number 20 divided the numbers $[m^5+5m^3+10m^2+4m]$, $[m^5+5m^3-10m^2+4m]$ and $[m^5-5m^3+4m]$.

Proof. By using Example 1, since the dimension of a vector space is an integer number, so the proof is complete. □

References

[1] M.R. Darafsheh, N.S. Poursalavati, *Orthogonal basis of the symmetry classes of tensors associated with the direct product of permutation groups*, Pure Mathematics and Applicans, 10 (1999), pp. 241–248.

[2] M.R. Darafsheh, N.S. Poursalavati, *On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups*, SUT Journal of Mathematics, 37 (2001), pp. 1–17.

[3] R. Freese, *Inequalities for generalized matrix functions based on arbitrary characters*, Linear Algebra, 7 (1973), pp. 337–345.

[4] R.R. Holmes, *Orthogonal bases of symmetrized tensor spaces*, Linear and Multilinear Algebra, 39 (1995), pp. 241–243.

[5] G. James, M. Liebeck, *Representations and characters of groups*, Cambridge University Press, 1993.

[6] M. Marcus, *Finite Dimensional Multilinear Algebra, part I*, Marcel Dekker, 1973.

[7] R. Merris, *Multilinear Algebra*, Gordon and Brreach scince publishers, 1997.

[8] B.Y. Wang, M.P. Gonge, *The subspace and orthonormal bases of symmetry classes of tensors*, Linear and Multilinear Algebra, 30 (1991), pp. 195–204.