Heat Transfer

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Chapter 5

Transient Conduction

Today’s Topics

- The Transient Heat Transfer in Plane Wall
- The Transient Heat Transfer in Cylinder
- The Transient Heat Transfer in Sphere
- The Semi-Infinite Solid
The Transient Heat Transfer in Plane Wall

Exact Solution: \[ \theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x') \]
where \( C_n = \frac{4 \sin \zeta_n}{2 \zeta_n + \sin(2 \zeta_n)} \) and \( \zeta_n \)'s are the positive roots of \( \zeta_n \tan \zeta_n = Bi \)

First four roots of this equation are given in Appendix B.3.

Approximate Solution: For \( Fo > 0.2 \) we have: \( \theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x') \)
or \( \theta^* = \theta_0^* \cos(\zeta_1 x') \) where \( \theta_0^* = C_1 \exp(-\zeta_1^2 Fo) \)

Table 5.1 shows the values of \( C_1 \) and \( \zeta_1 \) for a range of Biot numbers

Total Energy Transfer: \[ Q = -[E(t) - E(0)] = -\int \rho c [T(x,t) - T_e] dV \]
By defining: \( Q_0 = \rho c V(T_i - T_e) \) we have:
\[ \frac{Q}{Q_0} = \int \frac{[T(x,t) - T_e]}{T_i - T_e} dV \]
where \( \theta^* = \theta_0^* \cos(\zeta_1 x') \)

Graphical representations of the one-term approximation are in Appendix D.
The Transient Heat Transfer in Plane Wall

Table 5.1: Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

<table>
<thead>
<tr>
<th>$B^2$</th>
<th>$\xi_1$ (rad)</th>
<th>$C_1$</th>
<th>$\xi_2$ (rad)</th>
<th>$C_2$</th>
<th>$\xi_3$ (rad)</th>
<th>$C_3$</th>
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<td>0.01</td>
<td>0.0098</td>
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<td>1.0060</td>
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Figure D.1: Midplane temperature as a function of time for a plane wall of thickness 2L [1]. Used with permission.
The Transient Heat Transfer in Plane Wall

**FIGURE D.2** Temperature distribution in a plane wall of thickness $2L$ [1]. Used with permission.

The Transient Heat Transfer in Plane Wall

**FIGURE D.3** Internal energy change as a function of time for a plane wall of thickness $2L$ [2]. Adapted with permission.
The Transient Heat Transfer in Cylinder

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad 0 \leq r \leq r_o, \quad t > 0
\]

\[
\begin{align*}
\frac{\partial T}{\partial r} & = 0 \\
-k \frac{\partial T}{\partial r} & = h(T(r, t) - T_w) \\
T(r, 0) & = T_i
\end{align*}
\]

\[
\phi'(r^*, 0) = 1
\]

\[
T = T(r, t; T_i, T_w, r_o, k, \alpha, h)
\]

\[
0^* = \phi' (r^*, Fo, Bi), \quad Bi = \frac{hr_o}{k}
\]

\[
r^* = r / r_o, \quad \phi^* = \frac{\partial}{\partial r} \frac{T - T_w}{T_i - T_w}, \quad r^* = \frac{\alpha t}{r_o^2} = Fo
\]

\[
F_o = \frac{\alpha t}{r_o^2}, \quad Bi = \frac{hr_o}{k}, \quad r^* = \frac{r}{r_o}
\]

Exact Solution: \[\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_n(\zeta_n r^*)\]

where \[C_n = \frac{2}{\zeta_n J_n'(\zeta_n)} \quad \text{and} \quad \zeta_n's \quad \text{are the positive roots of} \quad \frac{J_n(\zeta_n)}{J_n'(\zeta_n)} = Bi\]

First four roots of this equation are given in Appendix B.4.

Approximate Solution: For \(Fo > 0.2\) we have: \[\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_1(\zeta_1 r^*)\]

or \[\theta^* = \theta_0^* J_0(\zeta_1 r^*) \quad \text{where} \quad \theta_0^* = C_1 \exp(-\zeta_1^2 Fo)\]

Table 5.1 shows the values of \(C_1\) and \(\zeta_1\) for a rang of Biot numbers.

Total Energy Transfer: \[\frac{Q}{Q_0} = 1 - \frac{2 \theta_0^*}{\zeta_1} J_1(\zeta_1)\]

Graphical representations of the one-term approximation are in Appendix D.
The Semi-Infinite Solid

The semi-infinite solid extends to infinity in all but one direction, it is characterized by a single identifiable surface. If a sudden change of conditions is imposed at this surface, transient, one-dimensional conduction will occur within the solid.

The similarity solution for case 1 (constant surface temperature):

\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad x > 0, \quad t > 0
\]

\[
\eta = \frac{x}{(4\alpha t)^{1/2}} = \text{Similarity variable}
\]

Case (1) \( T(x, 0) = T_i \)

Case (2) \( T(x, 0) = T_i \), \( -k \frac{\partial T}{\partial x} \bigg|_{x=0} = q_i' \)

Case (3) \( T(x, 0) = T_i \), \( -k \frac{\partial T}{\partial x} \bigg|_{x=0} = h(T_m - T_0), \quad \delta \)

Substituting in equation (1):

\[
\frac{\partial^2 T}{\partial \eta^2} = \frac{1}{d\eta} \frac{\partial T}{\partial \eta} \frac{d\eta}{\partial \eta} = \frac{1}{4\alpha t} \frac{d^2 T}{d\eta^2}
\]

\[
\frac{\partial T}{\partial t} = \frac{dT}{d\eta} \frac{\partial \eta}{\partial t} = \frac{x}{2t(4\alpha t)^{1/2}} \frac{dT}{d\eta}
\]

\[
\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}
\]

\[
T(\eta = 0) = T_i
\]

\[
T(\eta \to \infty) = T_i
\]
The Semi-Infinite Solid

\[
\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta} \quad \frac{d}{d\eta} \left( \frac{dT}{d\eta} \right) = -2\eta \frac{dT}{d\eta} \quad \frac{d(dT / d\eta)}{dT / d\eta} = -2\eta d\eta
\]

\[
\frac{dT}{d\eta} = C_1 \exp(-\eta^2) \quad T = C_1 \int_0^\eta \exp(-u^2) du + C_2
\]

\[
T_i = C_1 \int_0^\infty \exp(-u^2) du + C_2 \quad T_i = C_1 \pi^{1/2} / 2 + C_2 \quad \left\{ \begin{array}{l}
C_i = T_i \\
C_i = \frac{2(T_i - T)}{\pi^{1/2}}
\end{array} \right.
\]

\[
\frac{T(x,t) - T_i}{T_i - T} = \frac{2}{\pi^{1/2}} \int_0^\eta \exp(-u^2) du = \text{erf} \eta = \text{erf} \left( \frac{x}{(\pi \alpha t)^{1/2}} \right)
\]

\[
q_i^* = -k \frac{\partial T}{\partial x} \bigg|_{x=0} = -k \frac{dT}{d\eta} \bigg|_{\eta=0} = -k(T_i - T) \frac{\partial \eta}{\partial x} \bigg|_{\eta=0} = \frac{k(T_i - T)}{(\pi \alpha t)^{1/2}}
\]

The Semi-Infinite Solid

Case 2 (Constant Surface Heat Flux)

\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad x > 0 \quad t > 0
\]

\[
T(x,0) = T_i \quad T(x,t) - T_i = \frac{2q_i^*(at/\pi)^{1/2}}{k} \exp \left( -\frac{x^2}{4at} \right) - \frac{q_i^* x}{k} \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right)
\]

Case 3 (Surface Convection)

\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad x > 0 \quad t > 0
\]

\[
T(x,0) = T_i \quad T(x,t) - T_i = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) - \exp \left( \frac{h x}{k} + \frac{h' \alpha t}{k^2} \right) \left[ \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) - \exp \left( \frac{h x}{k} + \frac{h' \alpha t}{k^2} \right) \right]
\]
The Semi-Infinite Solid

\[ T_{x,A} = T_{x,B} = T_s \]

\[ q^*_{x,A} = q^*_{x,B} = -\frac{k(T_s - T_{x,A})}{(\pi \alpha_M T)^{1/2}} = \frac{k(T_s - T_{x,B})}{(\pi \alpha_M T)^{1/2}} \]

\[ T_t = \frac{(kpc)^{1/2}T_{x,A} + (kpc)^{1/2}T_{x,B}}{(kpc)^{1/2} + (kpc)^{1/2}} \]