CHAPTER 2
INTRODUCTION TO SPREAD SPECTRUM SYSTEMS

Problem 2-1

The received signal is given by (2-32). The power divider reduces the amplitude in each arm of the receiver by $2^{-1/2}$. Thus, with $T_d = \hat{T}_d = 0$ and $\phi = 0$,

$$x(t) = \sqrt{P}\, d(t) \{ c_1^2(t - T_c/2)\sqrt{2}\cos^2\left(\frac{\pi t}{T_c}\right) \cos(\omega_0 t) \times 2\cos[(\omega_0 t + \omega_{IF})\, t]$$

$$- \sqrt{2}\, c_1(t - T_c/2) c_2(t) \sin\left(\frac{\pi t}{T_c}\right) \cos\left(\frac{\pi t}{T_c}\right) \sin(\omega_0 t) \times 2\cos[(\omega_0 t + \omega_{IF})\, t] \}$$

and

$$y(t) = -\sqrt{P}\, d(t) \{ \sqrt{2}\, c_1(t - T_c/2) c_2(t) \cos\left(\frac{\pi t}{T_c}\right) \sin\left(\frac{\pi t}{T_c}\right) \cos(\omega_0 t) \times 2\sin[(\omega_0 t + \omega_{IF})\, t]$$

$$+ \sqrt{2}\, c_2^2(t) \sin^2\left(\frac{\pi t}{T_c}\right) \sin(\omega_0 t) \times 2\sin[(\omega_0 t + \omega_{IF})\, t] \}$$

The sum frequency terms are eliminated by the bandpass filter. The difference frequency term in the second term of $x(\mathbf{t})$ is cancelled by the difference frequency component of the first term of $y(t)$. Thus,

$$[x(t) + y(t)]_{\text{diff. freq.}} = \sqrt{2P}\, d(t) \left\{ c_1^2(t) \cos^2\left(\frac{\pi t}{T_c}\right) \cos(\omega_{IF} t) + c_2^2(t) \sin^2\left(\frac{\pi t}{T_c}\right) \cos(\omega_{IF} t) \right\}$$

Since $c_1^2(t) = c_2^2(t) = 1$, this reduces to

$$[x(t) + y(t)]_{\text{diff. freq.}} = \sqrt{2P}\, d(t) \cos(\omega_{IF} t)$$

Problem 2-2

(a) The bit error probability for the continuous noise jammer is given by

$$P_b = \frac{1}{2} e^{-E_b/N_J}$$

The pulse jammer turns on with a noise power spectral density of N_J/ρ for a franction ρ of the time. When the jammer is off, no errors are made assuming thermal noise is negligible. The average error probability is therefore

$$\overline{P_b} = \frac{\rho}{2} e^{-\rho E_b/N_J} + (1 - \rho)\times 0$$

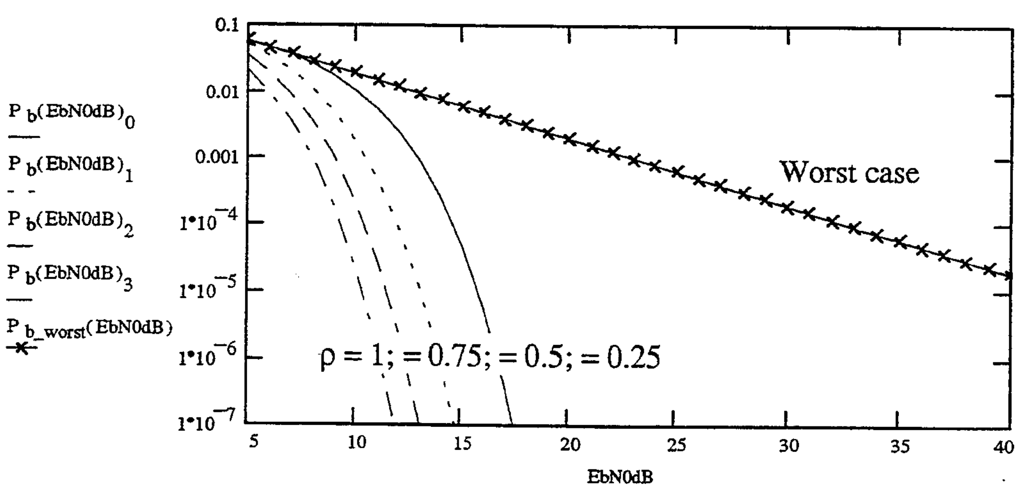
Taking the derivative with respect to ρ and setting equal to zero, we obtain

$$\rho_{opt} = \frac{1}{E_b/N_J}$$

This optimum ρ is valid only for $E_b/N_J \geq 1$. The worst case error probability is

$$\langle P_b \rangle = \begin{cases} \frac{1}{2}\exp\left(-\frac{E_b}{N_J}\right), & E_b/N_J < 1 \\ \frac{e^{-1}}{2(E_b/N_J)}, & E_b/N_J \geq 1 \end{cases}$$

(b) The plot is given below.



Problem 2-3

(a) The despreading operation in the receiver will spread the narrowband noise power over a bandwidth equal to the spreading waveform bandwidth. Assume that the jammer bandwidth is very narrow. Then the jammer power spectral density (one sided) at the output of the despreader is

$$S_J(f) = JT_c \text{sinc}^2[(f - f_0)T_c]$$

and its power spectral density near $f = f_0$ is

$$S_J(f_0) = JT_c$$

The jammer is pulsed with duty factor ρ. Thus the jammer power spectral density when on is

$$S_J(f_0) = JT_c/\rho$$

and the bit error probability (see Prob. 2-2) is

$$\overline{P_b} = \frac{\rho}{2}\exp\left(-\frac{\rho E_b}{JT_c}\right)$$

To find the optimum ρ, differentiate with respect to ρ and set the result to zero, giving

$$\rho_{\text{opt}} = \frac{1}{E_b/JT_c}, \quad \frac{E_b}{JT_c} \geq 1$$

Therefore,

$$\overline{P}_{b_{\text{worst case}}} = \begin{cases} \frac{1}{2}\exp\left(-\frac{E_b}{JT_c}\right), & \frac{E_b}{JT_c} < 1 \\ \frac{e^{-1}}{2(E_b/JT_c)}, & \frac{E_b}{JT_c} \geq 1 \end{cases}$$

Assume that the jammer for the non-spread case of Problem 2-2 uses a bandwidth equal to the data rate. Thus $N_J = JT$ where T is the data bit duration. Solve for J and substitute into the above expression for P_b to give

$$\overline{P}_{b_{\text{worst case}}} = \begin{cases} \frac{1}{2}\exp\left(-\frac{E_b}{JT_c/T}\right), & \frac{E_b}{JT_c/T} < 1 \\ \frac{e^{-1}}{2(E_b/JT_c/T)}, & \frac{E_b}{JT_c/T} \geq 1 \end{cases}$$

(b) Since the spreading code rate is 10 times the data rate, $T_c/T = 0.1$. Comparing this result with a similar result in Problem 2-2 shows that the processing gain is 10 dB.
(c) The optimum duty factors for the non-spread and spread systems are

$$\rho_{\text{optimum nonspread}} = \frac{N_J}{E_b}$$

and

$$\rho_{\text{optimum spread}} = \left(\frac{N_J}{E_b}\right)\left(\frac{T_c}{T}\right) \doteq 0.1\left(\frac{N_J}{E_b}\right)$$

The plot for this problem is the same as for Problem 2-2 except that the abscissa is relabeled $(E_b/N_J)_{\text{dB}}$ - 10 dB.

Problem 2-4

The processing gain is calculated using the same steps leading to (2-23). With the jammer center frequency at f_J, (2-21) becomes

$$J_0 = \frac{1}{2}JT_c\left[\int_{-\left(f_0+\frac{1}{2T}\right)}^{-f_0+\frac{1}{2T}} \text{sinc}^2[(f+f_J)T_c]\,df + \int_{f_0-\frac{1}{2T}}^{f_0+\frac{1}{2T}} \text{sinc}^2[(f-f_J)T_c]\,df\right]$$

Since the despreading code rate is 100 times the data rate, the sinc-functions can be approximated as constant over the ranges of integration. This gives

$$J_0 \approx J\frac{T_c}{T}\text{sinc}^2[(f_J-f_0)T_c]$$

The system processing gain is

$$\frac{J}{J_0} = \frac{T}{T_c\text{sinc}^2[(f_J-f_0)]} = \frac{100}{\text{sinc}^2[(f_J-f_0)]}$$

Problem 2-5

The power spectrum of the baseband MSK spreading waveform is

$$S_{MSK}(f) = \frac{8T_c[1 + \cos(4\pi f T_c)]}{\pi^2(1 - 16T_c^2 f^2)^2}$$

where unity power is assumed. The null-to-null bandwidth of this spectrum is 3/4 that of a BPSK with the same chip rate. Thus, for the MSK system, the same bandwidth is achieved using 1.333 times the chip rate of the equivalent BPSK system. Therefore

$$S_{MSK}(f) = \frac{8(0.75T_c)\{1 + \cos[(4\pi f)(0.75T_c)]\}}{\pi^2[1 - 16(0.75T_c)^2 f^2]^2}$$

where $T_c = 0.01T$. The power spectrum of the received jammer tone is convolved with $S_{MSK}(f)$ in the despreading operation so that the data demodulator input power is

$$J_0 = \frac{1}{2}J\left[\int_{-\left(f_0 + \frac{1}{2T}\right)}^{-f_0 + \frac{1}{2T}} S_{MSK}(f + f_J)\,df + \int_{f_0 - \frac{1}{2T}}^{f_0 + \frac{1}{2T}} S_{MSK}(f - f_J)\,df\right]$$

For T/T_c large, the power spectral density under each integral is approximately constant so that the system processing gain becomes

$$\frac{J}{J_0} = \frac{T}{T_c}\frac{\pi^2}{6}\frac{[1 - 9T_c^2(f_J - f_0)^2]^2}{1 + \cos[3\pi(f_J - f_0)T_c]}$$

Problem 2-6

From the shift register block diagram it follows that $c(t)$ has period $7T_c$ and one cycle is -1, -1, -1, 1, -1, 1, 1. One period of the autocorrelation function can be expressed as

$$R_c(\tau) = \left(1 + \frac{1}{7}\right)\Lambda\left(\frac{\tau}{T_c}\right) - \frac{1}{7} \quad |\tau| \le 7T_c$$

Using periodicity and a table of Fourier transforms, the power spectral density is

$$S_c(f) = \frac{1}{7T_c}\sum_{n=-\infty}^{\infty}\frac{8}{7}T_c \operatorname{sinc}^2(nf_0T_c)\,\delta(f - nf_0) - \frac{1}{7}\delta(f) = \sum_{n=-\infty}^{\infty} X_n\delta(f - nf_c)$$

where

$$X_n = \begin{cases} \frac{1}{49}, & n = 0 \\ \frac{8}{49}\,\mathrm{sinc}^2(n/7), & n \neq 0 \end{cases}$$

The data modulated IF signal is

$$y(t) = \sqrt{2P}A_m(t)\cos[\omega_0 t + \theta_m(t)]$$

where $A_m(t)$ and $\theta_m(t)$ are amplitude and phase modulations. Let $S_y(f)$ denote the power spectral density of $y(t)$. Then the power spectral density of the transmitted signal is

$$S_z(f) = S_y(f)*S_c(f) = \sum_{n=-\infty}^{\infty} X_n S_y(f - n/7T_c)$$

Problem 2-7

(a) For any t, say t_0, the mean of the random process is

$$\mathrm{E}[x(t_0)] = \frac{1}{4}(1) + \frac{1}{4}(-2) + \frac{1}{4}\sin(\pi t_0) + \frac{1}{4}\cos(\pi t_0) = -\frac{1}{4} + \frac{1}{4}(\sin \pi t_0 + \cos \pi t_0)$$

Since this is a function of the time instant chosen, the process is not stationary.
(b) By definition,

$$R_X(t_1, t_2) = \mathrm{E}[x(t_1)x(t_2)] = \frac{1}{4}(2) + \frac{1}{4}(-2)(-2) + \frac{1}{4}\sin(\pi t_1)\sin(\pi t_2) + \frac{1}{4}\cos(\pi t_1)\cos(\pi t_2)$$

$$= \frac{5}{4} + \frac{1}{4}\cos[\pi(t_1 - t_2)]$$

Problem 2-8

(a) Since $x(t)$ and $y(t)$ are independent, the probability density function (pdf) of their product is the product of their respective pdf's. Since they are both stationary, their individual pdfsare independent of time origin. Therefore, the product of their pdf's, or the pdf of $z(t)$, is independent of time origin.
(b) To get the power spectrum of $z(t)$, consider the autocorrelation function

$$R_Z(t_1, t_2) = \mathrm{E}[z(t_1)\,z(t_2)] = \mathrm{E}[x(t_1)y(t_1)x(t_2)y(t_2)] = \mathrm{E}[x(t_1)x(t_2)]E[y(t_2)y(t_2)] = R_X(\tau)R_Y(\tau)$$

Hence

$$S_Z(f) = S_X(f) * S_Y(f)$$

Problem 2-9

By definition, the autocorrelation function is

$$R_s(t_1, t_2) = \mathrm{E}[s(t_1, \underline{a}, T)s(t_2, \underline{a}, T)]$$

where the expectation is over $\underline{a}$ and T. For $|t_1 - t_2| > T_c$ the expectation over $\underline{a}$ gives zero, since t_1 and t_2 fall within different symbol intervals. For $|t_1 - t_2| \leq T_c$, we obtain

$$R_s(t_1, t_2) = \mathrm{E}\left[\sum_{n=-\infty}^{\infty} a_n p(t_1 + T - nT_c) \sum_{m=-\infty}^{\infty} a_m p(t_2 + T - mT_c)\right]$$

$$= \mathrm{E}\left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m p(t_1 + T - nT_c)p(t_2 + T - mT_c)\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathrm{E}[a_n a_m]\mathrm{E}[p(t_1 + T - nT_c)p(t_2 + T - mT_c)]$$

For $n \neq m$, $\mathrm{E}[a_n a_m] = 0$ which gives

$$R_s(t_1, t_2) = \sum_{m=-\infty}^{\infty} \mathrm{E}[p(t_1 + T - nT_c)p(t_2 + T - nT_c)]$$

For the waveform given, we are interested the value under the summation only for $0 \leq T \leq T_c$. Thus, there are only two terms of the summation of interest. Denote these by n_1 and $n_1 + 1$. Also, let $t_2 = t_1 + \tau$ with $0 \leq \tau \leq T_c$. The value of n_1 is such that $-T_c \leq n_1 T_c - t_1 \leq 0$. Then

$$R_s(t_1, t_1 + \tau) = \frac{1}{T_c}\int_0^{T_c} p(t_1 + T - n_1 T_c)p(t_1 + \tau + T - n_1 T_c)dT$$

$$+ \frac{1}{T_c}\int_0^{T_c} p[t_1 + T - (n_1 + 1)T_c]p[t_1 + \tau + T - (n_1 + 1)T_c]dT$$

If $\tau \leq t_1 - n_1 T_c$, this can be simplified to

$$R_s(t_1, t_1 + \tau) = \frac{1}{2\pi}\int_{2\pi\tau/T_c}^{2\pi} \sin(x)\sin(x + 2\pi\tau/T_c)dx$$

$$= \frac{1}{2}\left(1 - \frac{\tau}{T_c}\right)\cos(2\pi\tau/T_c) - \frac{1}{8\pi}\left[\sin(2\pi\tau/T_c) - \sin(6\pi\tau/T_c)\right]$$

If $\tau > t_1 - n_1 T_c$, the result is identical. Since the autocorrelation function is even, it follows that

$$R_s(\tau) = \frac{1}{2}\left(1 - \frac{\tau}{T_c}\right)\cos(2\pi\tau/T_c) - \frac{1}{8\pi}\left[\sin(2\pi|\tau|/T_c) - \sin(6\pi|\tau|/T_c)\right],\ -T_c \le \tau \le T_c$$

with $R_s(\tau + T_c) = R_s(\tau)$. To obtain the power spectral density, take the Fourier transform.

Problem 2-10

(a) This is slow frequency hop because the data rate is 100 times faster than the hop rate. The FSK signals will be orthogonal if the hop spacing is a multiple of 2×10^6 Hz. Because the synthesizer is noncoherent hop-to-hop, and because of the slow frequency hop, the transmit signal power spectral density is the sum of the frequency translations of the power spectral density of the data modulated carrier. All frequency translations have equal weight if all synthesizer frequencies are equally likely. Let the power spectral density of the data modulated carrier be

$$S_d(f) = S_0(f - f_0) + S_0(f + f_0)$$

where f_0 is an arbitrary center frequency. The power spectral density of the transmitted signal is

$$S_t(f) = \frac{1}{64}\sum_{m=0}^{63}\left[S_0(f - f_0 - m\Delta f) + S_0(f + f_0 + m\Delta f)\right]$$

where $\Delta f = 2$ MHz.
(b) The error probability for binary noncoherent FSK in AWGN is

$$P_b = \frac{1}{2}e^{-E_b/2N_0}$$

Assume that the interference is due solely to a partial band jammer that jams a fraction ρ of the band with one-sided power spectral density $N_J = J/\rho B$ where J is the total power of the jammer and B is the transmission bandwidth. The average bit error probability is

$$\overline{P_b} = \frac{1}{2}\rho \exp\left(-\frac{E_b \rho B}{2J}\right) = \frac{1}{2}\rho \exp\left(-\frac{1}{2}\frac{P}{J}\frac{B}{R}\rho\right)$$

where P is the average transmitter power and R is the data rate. Differentiating this with respect to ρ and setting the result equal to zero, we find the optimum ρ to be

$$\rho_{\text{optimum}} = 2\frac{J}{P}\frac{R}{B}$$

In the case at hand, $R = 10^6$ bps and $B \cong 128$ MHz, so

$$\rho_{\text{optimum}} = \frac{1}{64}\frac{J}{P}$$

Note that the optimum number of bands to jam is dependent on J/P.

Problem 2-11

This is fast frequency hop since the hop rate is 100 times the data rate. Separation between hop frequencies is set by spectral widening due to the hop itself. The hop spacing must be at least 1 MHz. At any one hop frequency, the power spectral density is approximately

$$S_0(f) = \frac{PT_c}{2}\left\{\text{sinc}^2[(f - f_0)T_c] + \text{sinc}^2[(f + f_0)T_c]\right\}$$

The transmitter uses 64 equally likely hop frequencies spaced by $\Delta f = 1$ MHz. Thus, the transmit signal power spectral density is

$$S_t(f) = \frac{PT_c}{128}\sum_{m=0}^{63}\left\{\text{sinc}^2[(f - f_0 - m\Delta f)T_c] + \text{sinc}^2[(f + f_0 + m\Delta f)T_c]\right\}$$

Note that arguments can be made for other hop spacings. For example, $\Delta f = 2$ MHz would make the spectrum more nearly non-overlapping.

Problem 2-12

Let the noise power spectral density at the input to the despreading mixer be $S_N(f)$. The power spectral density of the reference input to the despreading mixer is

$$S_R(f) = T_c\left\{\text{sinc}^2[(f - f_0)T_c] + \text{sinc}^2[(f + f_0)T_c]\right\}$$

Since the noise and the spreading code are independent, the power spectral density at the despreading mixer output is the convolution of these two separate spectra. Let $f_0 = f_1 + f_2$ where f_1 is the input center frequency and f_2 is the IF frequency. Due to the large processing gain, we are only interested in the magnitude of the convolution at $f = \pm f_2$. This gives the approximate result

$$S_{IF}(\pm f_2) \cong \frac{N_0 T_c}{2} \int_{-B/2}^{B/2} \operatorname{sinc}^2(T_c u)\, du$$

For $B = 2/T_c$,

$$S_{IF}(\pm f_2) \cong 0.903 \frac{N_0}{2}$$

For $B = 3/T_c$ and $B = 3/T_c$, the results are $0.931 N_0/2$ and $0.950 N_0/2$, respectively.

Problem 2-13

The transmitted waveform is

$$x(t) = \sqrt{2P}\, d(t - T_0) c_1(t - T_1) c_2(t - \tau - T_1) \cos(\omega_0 t + \phi)$$

where T_0, T_1, and ϕ are random delays and phase included to make the processes stationary. c_1 and c_2 use the same random delay because they are clock synchronous. The autocorrelation function of the transmitted waveform is

$$\begin{aligned} R_X(\lambda) &= 2P E_{d, T_0}\{d(t - T_0) d(t - \lambda\, T_0)\} \\ &\times E_{c_1, c_2, T_1}\{c_1(t - T_1) c_1(t - \lambda - T_1) c_2(t - \tau - T_1) c_2(t - \tau - \lambda - T_1)\} \\ &\times E_\phi\{\cos(\omega_0 t + \phi) \cos[(\omega_0(t - \lambda) + \phi)]\} \\ &= P \Lambda(\lambda / T_d) R_{c_1 c_2}(\lambda) \cos(\omega_0 \lambda) \end{aligned}$$

where

$$\Lambda(u) = \begin{cases} 1 - |u|, & |u| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and $R_{c1c2}(\lambda)$ is the autocorrelation function of the product of c_1 and c_2. This product can be viewed as a pair of time-interleaved pulse trains, the first with pulse width τ and the second with pulse width $T_c - \tau$. Both have periods T_c. The resultant autocorrelation function is

$$R_{c_1c_2}(\lambda) = \frac{\tau}{T_c}R_1(\lambda) + (1 - \tau/T_c)R_2(\lambda)$$

where $R_1(\lambda) = \Lambda(\lambda/\tau)$ and $R_2(\lambda) = \Lambda[\lambda/(T_c - \tau)]$. Thus, the complete autocorrelation function is

$$R_X(\lambda) = P\Lambda\left(\frac{\lambda}{T_d}\right)\left[\frac{\tau}{T_c}R_1(\lambda) + \left(1 - \frac{\tau}{T_c}\right)R_2(\lambda)\right]\cos(\omega_0\lambda)$$

The power spectral density is the Fourier transform of this.

Problem 2-14

During each signaling interval, a completely generic BPSK spread spectrum communication system would transmit

$$r_n(t) = s_m(t - nT_d)c(t),\ nT_d \le t \le (n+1)T_d,\ -\infty < m < \infty,\ m = 1,2,\cdots,M$$

where m is the message index and $s_m(t)$ is the data modulated signal waveform. The BPSK spreading waveform is $c(t)$ (it takes on the values ± 1 in T_c-second intervals). The minimum-probability-of-error receiver can be implemented in correlator or matched filter form with a correlator (matched filter) branch for each spread $s_m(t)$ followed by logic circuitry to choose the largest. The locally acquired spreading waveform can be multiplied with the incoming signal in each correlator branch or, equivalently, before the correlation with each $s_m(t)$.

Problem 2-15

The received signal is

$$r(t) = \sqrt{2P}\,d(t)c(t)\cos(\omega_0 t + \phi) + \sqrt{2J}\cos(\omega_0 t + \phi)$$

since the jammer is assumed coherent with the modulated signal. The integrator input is

$$[r(t)c(t)\times 2\cos(\omega_0 t + \phi)]_{LP} = \sqrt{2P}\left[d(t) + \sqrt{\frac{J}{P}}\,c(t)\right]$$

and the integrator output is

$$\sqrt{2P}\left[\pm 1 + \sqrt{\frac{J}{P}}\,\frac{1}{T}\int_0^T c(t)\,dt\right]$$

Write the spreading waveform as

$$c(t) = \sum_{n=-\infty}^{\infty} c_n p(t - nT_c)$$

where $c_n = \pm 1$ and $p(t)$ is a unit-amplitude square pulse of duration T_c. The integral then becomes

$$I = \frac{1}{N}\sum_{n=0}^{N} c_n$$

If the choice of the value of c_n is purely random, this sum is a binomially distributed random variable with distribution

$$P(I = 2k - N) = \binom{N}{k}\frac{1}{2^N},\ k = 0, 1, \cdots, N$$

Consider the normalized integrator output

$$X = \sqrt{\frac{J}{P}}\,\frac{I}{N}$$

It has probability distribution

$$P\left(IX = \frac{2k - N}{N}\sqrt{\frac{J}{P}}\right) = \binom{N}{k}\frac{1}{2^N},\ k = 0, 1, \cdots, N$$

When $d(t) = -1$ is transmitted, and error is made whenever $X > 1$. Thus

$$\Pr[\text{error}\,|\,d(t) = -1] = \sum_{k=\ell}^{N}\binom{N}{k}\frac{1}{2^N}$$

where ℓ is the smallest k for which

$$\frac{2k - N}{N}\sqrt{\frac{J}{P}} > 1$$

An identical result is found for $d(t) = 1$ so that the above expression is the average probability of error. For typical results, let $N = 7$ and 15 and $J/P = 0, 1, 2, \ldots 10$ dB. The results for ℓ and the probability of error are given in the table below.

J/P, dB	$N = 7; \ell$	$N = 7; P_e$	$N = 15; \ell$	$N = 15; P_e$
0	7	0.00781	15	3.052×10^{-5}
1	7	0.00781	15	3.052×10^{-5}
2	7	0.00781	14	4.883×10^{-4}
3	6	0.06250	13	0.00369
4	6	0.06250	13	0.00369
5	6	0.06250	12	0.01758
6	6	0.06250	12	0.01758
7	6	0.06250	11	0.05923
8	5	0.22656	11	0.05923
9	5	0.22656	11	0.05923
10	5	0.22656	10	0.15088